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*A Method for Calculating Simultaneously all the Roots of an Equation.**

BY EMORY MCCLINTOCK.

The comprehensive method which forms the subject of this paper may be introduced best by practical illustrations, beginning with trinomials and proceeding later to equations in general. Let it be desired to learn approximately all the roots of the equation $x^6 = -1 - x$. In this case we may use the formula

$$x = \omega - \omega^2 a - \tfrac{3}{2}\omega^3 a^2 - \tfrac{8}{3}\omega^4 a^3 - \ldots, \tag{1}$$

where $a = -\frac{1}{6}$, and ω is any one of the sixth-roots of -1, viz.

m	ω_1^m	ω_2^m	ω_3^m	ω_4^m	ω_5^m	ω_6^m
1	$\sqrt{-1}$	$-\sqrt{-1}$	$\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{-1}$	$\frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{-1}$	$-\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{-1}$	$-\frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{-1}$
2	-1	-1	$\frac{1}{2}+\frac{1}{2}\sqrt{-3}$	$\frac{1}{2}-\frac{1}{2}\sqrt{-3}$	$\frac{1}{2}-\frac{1}{2}\sqrt{-3}$	$\frac{1}{2}+\frac{1}{2}\sqrt{-3}$
3	$-\sqrt{-1}$	$\sqrt{-1}$	$\sqrt{-1}$	$-\sqrt{-1}$	$\sqrt{-1}$	$-\sqrt{-1}$
4	1	1	$-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$	$-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$	$-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$	$-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$
5	$\sqrt{-1}$	$-\sqrt{-1}$	$-\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{-1}$	$-\frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{-1}$	$\frac{1}{2}\sqrt{3}+\frac{1}{2}\sqrt{-1}$	$\frac{1}{2}\sqrt{3}-\frac{1}{2}\sqrt{-1}$
6	-1	-1	-1	-1	-1	-1

By inserting in (1) the numerical value of a we have

$$x = \omega + \tfrac{1}{6}\omega^2 - \tfrac{1}{24}\omega^3 + \tfrac{1}{81}\omega^4 - \ldots.$$

For the several pairs of roots, or values of x, we have therefore, taking the values of ω as stated, and remembering that $\sqrt{3} = 1.732$ nearly,

$$x = -\tfrac{1}{6} \quad + \tfrac{1}{81} - \quad \ldots \pm \sqrt{-1}\,(1 + \tfrac{1}{24} \ldots) = -\tfrac{25}{162} \pm \tfrac{25}{24}\sqrt{-1}, \text{ nearly;}$$

$$x = \tfrac{1}{2}\sqrt{3} + \tfrac{1}{12} - \tfrac{1}{162} \ldots \pm \sqrt{-1}\,(\tfrac{1}{2} + \tfrac{1}{12}\sqrt{3} - \tfrac{1}{24} + \tfrac{1}{162}\sqrt{3} \ldots)$$
$$= .94 \pm .61\sqrt{-1}, \text{ nearly;}$$

$$x = -\tfrac{1}{2}\sqrt{3} + \tfrac{1}{12} - \tfrac{1}{162} \ldots \pm \sqrt{-1}\,(\tfrac{1}{2} - \tfrac{1}{12}\sqrt{3} - \tfrac{1}{24} - \tfrac{1}{162}\sqrt{3} \ldots)$$
$$= -.79 \pm .30\sqrt{-1}, \text{ nearly.}\dagger$$

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*Read before the American Mathematical Society on August 14 and October 27, 1894. The portion read on August 14 is indicated in the Society's Bulletin for October, 1894.

† This example has been employed by Spitzer (Allgemeine Auflösung der Zahlengleichungen, Wien, 1851) and by Jelinek (Die Auflösung der höheren numerischen Gleichungen, Leipzig, 1865), to illustrate

The formula (1) made use of is valid only for equations of the form $x^6 = -1 + 6ax$, where a is numerically less than the sixth-root of 5^{-5}. The same formula may be used, with the same restriction upon the value of a, for equations of the form $x^6 = 1 - 6ax$, by taking ω for any sixth-root of 1. It is a special case of a more general formula, applicable to all equations of the form $x^n = \omega^n + nax^{n-k}$ (that is to say, all trinomial equations) for which the series is convergent:

$$\begin{aligned} x = \omega + \omega^{1-k}a + \omega^{1-2k}(1-2k+n)\,a^2/2\,! \\ + \omega^{1-3k}(1-3k+n)(1-3k+2n)\,a^3/3\,! \\ + \omega^{1-4k}(1-4k+n)(1-4k+2n)(1-4k+3n)\,a^4/4\,! + \ldots . \end{aligned} \tag{2}$$

Here ω may have any value, but a^n must for convergency be smaller numerically than $k^{-k}(n-k)^{k-n}\omega^{nk}$ when n is positive. If n is negative, a^{-n} must for convergency be smaller numerically than $(-k)^k(k-n)^{n-k}\omega^{-nk}$. Nothing is gained by having n negative, since in that event we have only to multiply $x^n = \omega^n + nax^{n-k}$ by $x^{-n}\omega^{-n}$ to produce the positive form $x^{-n} = \omega^{-n} - nbx^{-k}$, where $b = a\omega^{-n}$.

In the interpretation of the trinomial formula (2), it is usually most convenient to reduce the given trinomial $x^n = \omega^n + nax^{n-k}$ to that form in which ω^n is 1 or -1, in which case ω means merely any n^{th} root of 1 or -1, as the case may be. If that is not done, we must interpret ω as equivalent to $c\xi$, where c is the n^{th} root of the numerical value of ω^n, and ξ is any n^{th} root of 1 or -1, according as ω^n is positive or negative. This remark will apply to other equations as well as to trinomials. For uniformity of illustration, the examples adduced will be of the form $\omega^n = 1$ or $\omega^n = -1$, to which form any equation $x^n = \pm c^n + f(x)$ is at once reduced by writing cx for x.

Since its first use by Newton, if Newton was its author, no discussion of numerical equations can be considered complete without introducing the celebrated equation $x^3 - 2x - 5 = 0$.* To this as it stands we can apply the trinomial formula (2) at once, but a more convergent series may be had by suppressing the term next to the last, by the transformation $x^{-1} = y - 2/15$, so that the equation to be solved becomes $3375\,y^3 - 180\,y - 659 = 0$. In this

the changes made by them in Horner's method to make it serve in the computation of imaginary roots. Such a calculation as theirs can only be done for one pair at a time, and that with considerable difficulty, after first assigning approximate locations.

*Employed after Newton for successive new methods by Lagrange, Fourier, Sturm, and Murphy, not to speak of other writers.

let $y = (659/3375)^{\frac{1}{3}} z = .580146\,z$,* so that $z^3 = 1 + 3az$, where $a = .0528206$. Here $\omega^3 = 1$, and by (2) we have

$$z = \omega + \omega^2 a - \tfrac{1}{3}\omega a^3 + \ldots.$$
$$= \omega + .05282\omega^2 - .00005\omega + \ldots. = .99995\omega + .05282\omega^2 \ldots.$$

The cube roots of 1 being 1 and $-\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$, the three values of z are therefore 1.05277 and $-.52639 \pm .47356\sqrt{-3}$, whence those of y are .60176 and $-.30088 \pm .47585\sqrt{-1}$, all correct to the last place. The real root only has heretofore been sought by the various writers who have dealt with this equation, so far as appears, except Murphy, who finds approximate values of the imaginary roots by two methods, both tedious.†

Murphy adds another example, occupying two or three pages, concerning which, writing in 1838, he says: "…. as this method may be said to be the only direct one known for obtaining a first notion of the magnitudes of the real and imaginary parts of the roots of equations, we have, therefore, developed it at length ….; we may add that the research of the impossible roots of equations has been generally overlooked in modern treatises of algebra."‡ The equation which he employs is $x^4 + x + 10 = 0$, and he finds eventually, for the four roots, $1.251 \pm 1.348\sqrt{-1}$ and $-1.251 \pm 1.282\sqrt{-1}$. If we take $x = 10^{\frac{1}{4}}y = 1.77828\,y$, the equation to be solved becomes $y^4 + 10^{-\frac{3}{4}}y + 1 = 0$, or $y^4 = -1 + 4ay$, where $a = -.044457$. Applying (2), and observing that $\omega^4 = -1$, $n = 4$, $k = 3$,

$$y = \omega + \omega^{-2}a - \tfrac{1}{2}\omega^{-5}a^2 + \ldots. = \omega - \omega^2 a - \tfrac{1}{2}\omega^3 a^2 + \ldots.$$
$$= \omega + .044457\,\omega^2 - .000988\,\omega^3 + \ldots.$$

The first term neglected is $-\frac{7}{8}\omega a^4$, so that the terms taken should be good to five places. When $\omega^4 = -1$, we have either $\omega = \sqrt{\tfrac{1}{2}}(1 \pm \sqrt{-1})$, $\omega^2 = \pm\sqrt{-1}$, $\omega^3 = \sqrt{\tfrac{1}{2}}(-1 \pm \sqrt{-1})$; or, $\omega = \sqrt{\tfrac{1}{2}}(-1 \pm \sqrt{-1})$, $\omega^2 = \mp\sqrt{-1}$, $\omega^3 = \sqrt{\tfrac{1}{2}}(1 \pm \sqrt{-1})$. Also, $\sqrt{\tfrac{1}{2}} = .707107$. Taking the first pair of values of ω, we have

$$y = .707107\,(1 + .000988) \pm .044457\sqrt{-1} \pm .707107\,(1 - .000988)\sqrt{-1}$$
$$= .70781 \pm .75086\sqrt{-1},$$

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*Decimal fractions which are only approximately true will be stated, for convenience, without any such qualifying phrase as "nearly."

† *Theory of Equations*, p. 124; pp. 135–138.

‡ Todhunter remarks that "there is no easy practical method of calculating the imaginary roots of equations at present known." Cayley, 1878, in the article "Equation" in the Encyclopædia Britannica, says: "Very little has been done in regard to the calculation of the imaginary roots of an equation by approximation; and the question is not here considered."

and similarly for the second pair,

$$y = -.70781 \pm .66195\sqrt{-1}.$$

The four values of $x = 1.77828y$ are therefore $x = 1.2587 \pm 1.3352\sqrt{-1}$ and $x = -1.2587 \pm 1.1771\sqrt{-1}$. These values are very closely correct, so that Murphy may possibly have made some error, though he says distinctly that his results must be taken as only a first approximation.

Enough has been said to illustrate the use of (2) when the given trinomial is fit—a phrase which will be used to express readiness for the application of the methods now brought forward so as to yield a convergent series. Murphy's quartic was found fit at once, as was the sextic first discussed. Newton's cubic was improved by a linear transformation; a process not usually available, when a trinomial is desired, for degrees above the third. Two questions therefore arise: what shall be done to improve a given cubic, and what shall be done when the degree is higher and the trinomial is unfit?

If the cubic $x^3 = \pm 1 + 3ax$ has $a^3 < \frac{1}{4}$,* it is fit, though a transformation may secure greater convergency. If $a^3 = \frac{1}{4}$, a being positive, there are two equal roots, and no transformation will avail. If $a^3 \geqq \frac{1}{4}$, and if a is negative, the transformation $x^{-1} = y \mp a$ will serve, though some other may serve better. If $a^3 > \frac{1}{4}$, and if a is positive, the roots of the cubic are all real, and the exhibition of all of them at once is impossible, since the formula (2) expressly contemplates, for cubics, two imaginary roots, corresponding to the two imaginary cube roots of unity employed. In this latter case the trinomial is radically unfit, and the disposition to be made of it may be considered along with that of unfit trinomials of higher degrees, as part of the second question.

Except for improvable cubics, as just explained, there is usually nothing better to be done with an unfit trinomial, as a trinomial, than to apply the trinomial formula (2) twice, securing $n-k$ roots by one operation, and the remaining k roots by the second; though sometimes the trinomial form may be abandoned advantageously. The original trinomial being $x^n = \omega^n + nax^{n-k}$, we now regard it, for the first operation, as $x^{n-k} = \omega_1^{n-k} + (n-k)a_1x^n$, where $\omega_1^{n-k} = -\omega^n/na$, and a_1 is the reciprocal of $(n-k)na$. More simply, if $x^n = \pm 1 + nax^{n-k}$, $x^{n-k} = (\mp 1 + x^n)/na$. That $n-k$ roots can now be produced by a convergent series is readily proved. By the supposition of unfitness,

* Numerical values always understood.

$a^n > k^{-k}(n-k)^{k-n}\omega^{nk}$, and we have also $a = n^{-1}(n-k)^{-1}a_1^{-1}$, $\omega^n = -na\omega_1^{n-k}$ $= a_1^{-1}\omega_1^{n-k}/(k-n)$. Hence $n^{-n}(n-k)^{-n}a_1^{-n} > k^{-k}(n-k)^{k-n}a_1^{-k}\omega_1^{k(n-k)}(k-n)^{-k}$, whence $a_1^{n-k} < n^{-n}(-k)^k\omega_1^{-k(n-k)}$, which satisfies the criterion of convergency first stated, it being observed that $n-k$ now takes the place of n, n that of $n-k$, and $-k$ that of k.

For example, let

$$4y^3 - 243y + 165 = 0. \tag{3}$$

Here $y^3 = -165/4 + 3(81/4)y$, and the numerical value of $(81/4)^3$ is not less than that of $2^{-2}1^{-1}(165/4)^2 = (165/8)^2$. We must therefore, since a is positive, employ two operations: one upon the equation $y = 165/243 + 4y^3/243$, and the second upon the equation $y^2 = 243/4 - 165y^{-1}/4$, the first yielding one real root, the second yielding the two remaining real roots. For the first, let $y = 55u/81$, so that the equation reduces to $u = 1 + 4.55^2.3^{-1}.81^{-3}u^3$ $= 1 + .00759u^3$. Here $a = .00759$, $n = 1$, $k = -2$, and from (2) we have

$$u = 1 + a + 3a^2 + \ldots\ldots = 1 + .00759 + .00017 = 1.00776.$$

Hence $y = .684$. For the second, let $y = \frac{1}{2}243^{\frac{1}{2}}v$, so that the equation reduces to $v^2 = 1 - 330(243)^{-\frac{3}{2}}v^{-1} = 1 - .08712v^{-1}$, or $v^2 = 1 + 2av^{-1}$, where $a = -.04356$. By (2), taking $n = 2$, $k = 3$,

$$v = \omega - \omega^{-2}.04356 - \omega^{-5}.00285 \ldots\ldots = .997\omega - .044.$$

As ω may be 1 or -1, v is .953 or -1.041; and as $y = 7.794v$, the remaining values of y are 7.43 and -8.11.

In this example we have illustrated not only the first operation, securing $n-k (= 1)$ root, but also the second operation, securing the remaining $k (= 2)$ roots. For the latter the formula used is equivalent to $x^k = na + \omega^n x^{k-n}$, obtained by multiplying both sides of the original trinomial $x^n = \omega^n + nax^{n-k}$ by x^{k-n}. For the simpler form $x^n = \pm 1 + nax^{n-k}$, we have $x^k = na \pm x^{k-n}$. That this transformation necessarily yields a convergent series when the original trinomial is unfit may be seen upon due substitution. For, putting $\omega^n = ka_1$ and $a = n^{-1}\omega_1^k$, we have, as the criterion of convergency of the derived equation $x^k = \omega_1^k + ka_1x^{k-n}$, the requirement $a_1^k < n^{-n}(k-n)^{n-k}\omega_1^{nk}$, and this is satisfied by substituting for ω^n and a their values in the known numerical inequality $a^n > k^{-k}(n-k)^{k-n}\omega^{nk}$.

Referring to the last example (3), we took first $n=1$. Whenever $n=1$ the trinomial equation $x^n=\omega^n+nax^{n-k}$ becomes $x=\omega+ax^m$, putting $k=1-m$. Then (2) becomes, for this special case,

$$x=\omega+\omega^m a+\omega^{2m-1}ma^2+\omega^{3m-2}m(m-1)a^3/2!+\ldots\ldots, \tag{4}$$

a series known to Euler and Lagrange, and immediately derivable from Lagrange's theorem,

$$f(x)=f(\omega)+\phi(\omega)f'(\omega).a+\frac{d}{d\omega}\{[\phi(\omega)]^2f'(\omega)\}.a^2/2!+\ldots\ldots, \tag{4*}$$

where the relation is $x=\omega+a\phi(x)$. The trinomial series (2) may be derived at once, as will be seen, from Lagrange's theorem (4*), by putting $x^n=u$, so that the relation is $u=\omega^n+nau^{1-k/n}$, and expanding $f(u)=u^{1/n}$ by means of (4*). This is however an afterthought, the series (2) having been discovered, without employing Lagrange's theorem, in the course of writing the preceding paper ("Theorems in the Calculus of Enlargement") together with and as a case of a more general series which applies to other as well as to trinomial equations.

The new theorems numbered (21) and (22) in the paper just referred to are, when $x^n=\omega^n+na\phi(x)$,

$$x=\omega+\omega^{1-n}\phi\omega.a+\mathrm{E}_0 0(\phi\mathrm{E}_0)^2\omega^{0-2n}(0-n).a^2/2! \\ +\mathrm{E}_0 0(\phi\mathrm{E}_0)^3\omega^{0-3n}(0-n)(0-2n).a^3/3!+\ldots\ldots, \tag{5}$$

$$x=\omega+\omega^{1-n}\phi\omega.a+\omega^{1-n}\frac{d}{d\omega}\omega^{1-n}(\phi\omega)^2.a^2/2! \\ +\left(\omega^{1-n}\frac{d}{d\omega}\right)^2\omega^{1-n}(\phi\omega)^3.a^3/3!+\ldots\ldots \tag{6}$$

Here the brackets about the letter following the functional sign ϕ are omitted for convenience. The two theorems (5) and (6) are of course identical, except as to the form of the coefficients, which have the same real value, each to each. The symbol 0 is equivalent to t, when $t=0$, and the symbol of operation E_0 is to be interpreted as that by which, when applied to any function of 0, say $f0$, the latter becomes the same function of $0+1$, say $f(0+1)$. Similarly, this operation being subject to the law of indices, $\mathrm{E}_0^h f0=f(0+h)$; and more generally, if $\phi\mathrm{E}_0=a\mathrm{E}_0^n+b\mathrm{E}_0^{n-1}+\ldots\ldots$, $\phi\mathrm{E}_0 f0=a\mathrm{E}_0^n f0+b\mathrm{E}_0^{n-1}f0+\ldots\ldots=af(0+n)+bf(0+n-1)+\ldots\ldots$ So also with powers of $\phi\mathrm{E}_0$; if for instance $\phi\mathrm{E}_0=\mathrm{E}_0^n+\mathrm{E}_0^m$, $(\phi\mathrm{E}_0)^2f0=(\mathrm{E}_0^{2n}+2\mathrm{E}_0^{n+m}+\mathrm{E}_0^{2m})f0=f(0+2n)+2f(0+n+m)+f(0+2m)$.

If, for example, in (5), $\phi x=x^{n-k}$, so that $\phi\mathrm{E}_0=\mathrm{E}_0^{n-k}$, the coefficient of a is $\omega^{1-n}\omega^{n-k}=\omega^{1-k}$, as in (2). Again, the coefficient of $a^2/2!$ is $\mathrm{E}_0 0\mathrm{E}_0^{2n-2k}\omega^{0-2n}(0-n)$

$= E_0 0\omega^{0-2k}(0+n-2k) = (0+1)\,\omega^{0+1-2k}(0+1+n-2k) = \omega^{1-2k}(1-2k+n)$, as in (2); and the other coefficients of (2) will be confirmed in like manner, so that (2) is a special case of (5). It is likewise a special case of the conjugate expression (6), which is derived from Lagrange's theorem (4*) by writing x^n, ω^n, and na, for x, ω, and a respectively, and taking $x = f(x^n) = (x^n)^{\frac{1}{n}}$. This use of Lagrange's theorem has escaped notice since 1768, when that theorem was published, and might have escaped notice much longer had it not been for the circumstance that the relation underlying (5), from the nature of its origin, is $x^n = \omega^n + na\phi x$ instead of $x = \omega + a\phi x$. Given this relation and this series (5), it is most natural to remark that ω must have n values, determined by the n^{th} roots of 1 or of -1.

The theorem (5) is itself a special case of the more general theorem, numbered (19) in the preceding paper,

$$fx = f\omega + fE_0 0\phi E_0\omega^{0-n}.a + fE_0 0\,(\phi E_0)^2\,\omega^{0-2n}(0-n).a^2/2! + \ldots, \tag{7}$$

from which

$$x^m = \omega^m + m\omega^{m-n}\phi\omega.a^* + m\,(\phi E_0)^2\omega^{0+m-2n}(0+m-n).a^2/2! + \ldots, \tag{8}$$

the given relation being still $x^n = \omega^n + na\phi x$. By means of this equation we can find approximations, in convergent cases, to the m^{th} powers of the several roots of a given equation $x^n = \omega^n + na\phi x$, by employment of the n^{th} roots of 1 or of -1. Apart from the case $m = 1$, however, I see no practical use for any other case than $m = -1$. In this case (8) becomes

$$x^{-1} = \omega^{-1} - \omega^{-1-n}\phi\omega.a - (\phi E_0)^2\,\omega^{0-1-2n}(0-1-n).a^2/2! - \ldots \tag{9}$$

For the trinomial case $x^n = \omega^n + nax^{n-k}$, this becomes

$$x^{-1} = \omega^{-1} - \omega^{-1-k}.a - \omega^{-1-2k}(-1-2k+n).a^2/2! - \ldots \tag{10}$$

Before proceeding to discuss the principal series (5), it will be well to consider finally the reciprocal series (9) and (10). It is obvious that in any of the trinomial examples already brought forward we might, with little additional difficulty, by the use of (10), obtain different approximations to the roots by means of their reciprocals. Take, for instance, the equation numbered (3), where $u = 1 + .00759\,u^3$. Here $n = 1$, $k = 2$, $a = .00759$, and from (10),

$$u^{-1} = 1 - a - 2a^2 \ldots = 1 - .00759 - .00012 = .99229.$$

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* Here $m\phi E_0\omega^{0+m-n} = m\omega^{m-n}\phi E_0\omega^0$, and $\phi E_0\omega^0 = \phi\omega$, by a known theorem.

This differs but 1 in the last place from the reciprocal of the value found for u, 1.00776, a matter of no importance considering that the result is desired to three places only. Cases may arise in which these reciprocal formulæ will be found valuable; and they will certainly be found of the highest value if and when means are discovered for distinguishing those cases in which the reciprocal approximation is the more correct. Little appears to be gained, on the average, by taking a mean between the values ascertained by the direct and reciprocal approximations; yet in half of all cases the reciprocal approximation must be the closer of the two. Probably in most cases both the direct and the reciprocal approximations will err on the same side of the true value; and as regards the contrary chance, it seems better to continue one approximation further than to go to the labor of computing both. At present, therefore, no practical advantage appears to be derived from the use of these reciprocal theorems, though further investigation may enable them to take the place, in suitable cases, of the direct. It will be found, on examination, and may readily be proved, that the same series is obtained by (5) or (6) for x, from $x^n = \omega^n + na\phi x$, as by (9) for $x = y^{-1}$ from $y^n = \omega^{-n} - na\omega^{-n} y^n \phi(y^{-1})$. The differential formulæ which correspond to (7), (8), and (9) as (6) corresponds to (5) are, respectively,

$$fx = f\omega + \omega^{1-n}\phi\omega f'\omega . a + \left(\omega^{1-n}\frac{d}{d\omega}\right)[\omega^{1-n}(\phi\omega)^2 f'\omega] . a^2/2! + \ldots , \quad (11)$$

$$x^m = \omega^m + m\omega^{m-n}\phi\omega . a + m\left(\omega^{1-n}\frac{d}{d\omega}\right)[\omega^{m-n}(\phi\omega)^2] . a^2/2! + \ldots , \quad (12)$$

$$x^{-1} = \omega^{-1} - \omega^{-1-n}\phi\omega . a - \left(\omega^{1-n}\frac{d}{d\omega}\right)[\omega^{-1-n}(\phi\omega)^2] . a^2/2! - \ldots \quad (13)$$

Returning to the general solution (5), for which hereafter the reader may at his option substitute (6) as equivalent, we have seen that when $\phi x = x^{n-k}$, that is, when the given equation $x^n = \omega^n + na\phi x$ is a trinomial, the general solution takes the form of (2). For all cases in which (2) is not available, and in fact for all cases whether (2) is available or not, the solution is equally general if $a = 1$. Again, since $\mathrm{E}_0 0 = 1$, the first zero in each term of (5) may be omitted. We shall therefore, now that we are passing beyond the consideration of trinomials as such, write the general solution of $x^n = \omega^n + n\phi x$ thus:

$$x = \omega + \omega^{1-n}\phi\omega + \tfrac{1}{2}(\phi\mathrm{E}_0)^2\omega^{0+1-2n}(0+1-n)$$
$$+ \tfrac{1}{6}(\phi\mathrm{E}_0)^3\omega^{0+1-3n}(0+1-n)(0+1-2n) + \ldots \quad (14)$$

Knowing this formula, we are usually enabled, almost at a glance, to determine the number of imaginary roots, the signs of the real roots, and often the signs of the real parts of the imaginary roots, of any given equation; and applying it, we can obtain with little difficulty approximations more or less exact to the values of all the roots. This means that Sturm's theorem, acknowledged hitherto to be the only complete solution of the problem of separation of roots, is no longer essential in the examination of numerical equations. To make this clear, and also to avoid the charge, brought by Fourier against Euler, of selecting easy examples, I shall shortly take up in order all of the illustrations, eleven in number, employed in Burnside and Panton's *Theory of Equations* in the section devoted to the application of Sturm's theorem.

Having an equation presented for solution, the first step is to inspect it, to see whether it is fit. If not, some linear transformation must be sought to render it fit. If that prove impracticable, the inventor of the equation may be suspected of introducing equal roots and known tests may be applied. (With coefficients taken at random, equal roots are not likely to appear.) If no equal roots are found, a suitable transformation is possible. No equation is certainly fit, however, unless all the roots can be found by convergent series.

I shall use the word "span" for the degree of an operation, represented by the letter n in $x^n = \omega^n + n\phi x$. Thus, for Jelinek's equation, $x^6 + x + 1 = 0$, the span is sextic, and the equation is solved with one span. For Newton's equation, $x^3 - 2x - 5 = 0$, there is but a single span, a cubic; and for Murphy's equation, $x^4 + x + 10 = 0$, likewise but one, a quartic. For the equation numbered (3), however, $4y^3 - 243y + 165 = 0$, two spans are needed, namely, from left to right, a quadratic and a linear span. Examining these trinomials, we see that in the first three the middle coefficient is small enough to permit a single span, while in the last it is too large. I say therefore that in the first three cases the "dominant" coefficients are the first and last, while in the last case all three coefficients are "dominant." A span stretches from one dominant to the next. There is therefore but one span for each of the first three equations, while two are indicated for the last. If the dominants which define any span have like signs, it is a "like span"; otherwise an "unlike span." The recognition of dominants is not always easy, but is often facilitated by some simple transformation. Once recognized, they disclose the nature of all the roots at a glance. If exceptions exist to this statement, they have yet to be discovered. Each span represents n roots. It will be remembered that ω^n is positive

for a span whose dominants have unlike signs, and *vice versa.* The following table, showing the number of different kinds of roots covered by each span, is based on the characteristics of the n^{th} roots of 1 or of -1:

Description of Span.	Real Roots.		Imaginary Roots.		
	Positive.	Negative.	Positive.	Negative.	Uncertain.
$n=1$, unlike signs:	1				
like:		1			
$n=2$, unlike:	1	1			
like:					2
$n=3$, unlike:	1			2	
like:		1	2		
$n=4$, unlike:	1	1			2
like:			2	2	
$n=5$, unlike:	1			2	2
like:		1	2		2
$n=6$, unlike:	1	1	2	2	
like:			2	2	2

Recurring to the examples given, we find for $x^6+x+1=0$, $n=6$, signs like, therefore six imaginary roots, two of them at least with real parts positive, two at least negative; for $x^3-2x-5=0$, $n=3$, signs unlike, one real positive, two imaginary with real parts negative; for $x^4+x+10=0$, $n=4$, like signs, two pairs of imaginary roots, the real parts having opposite signs; and for that numbered (3), $4y^3-243y+165=0$, first span, $n=2$, signs unlike, therefore two real roots of opposite signs, to which the second span, $n=1$, again unlike, adds another real positive.

A recognition of the dominants of any equation not only indicates the nature of the roots, but shows how, if desired, we can proceed first to the computation of the greatest or least root. It will be found that the first span of n_1 terms yields not only n_1 roots, but the n_1 greatest of the roots; that the next span of n_2 terms yields the n_2 roots next greater in size, and so on, till the last span of n_k terms, which must yield the n_k roots of smallest size. In this statement imaginary roots rank according to their moduli. For, let a, b, c be the numerical values of successive dominants, always counting from left to right, with the highest power of the unknown quantity, as usual, on the left. The point to be shown is that the roots covered by the span from a to b, of k degrees, are all larger than the largest of those covered by the span from b to c, of l degrees.

That b shall be a dominant requires that b^{k+l} shall be decidedly larger than $a^l c^k$; let this be assumed. To put these spans successively in the form $v^n = \pm 1 + n\phi v$, we first put $n = k$, $x = (b/a)^{1/k} v_1$, and afterwards $n = l$, $x = (c/b)^{1/l} v_2$. As regards the first span, the series is convergent for the k values of v_1 which are nearest to unity; and as regards the second span, the series is convergent for the l values of v_2 which are nearest to unity. Since $(b/a)^{1/k}$ is decidedly larger than $(c/b)^{1/l}$, the values of x corresponding to the k values of v_1 must be larger than the largest of those which correspond to the l values of v_2. That any one root is not presented in each of two adjacent spans, is immediately assented to when we reflect that if it were, some other root must fail of presentation at all. That this intuitive assent is correct may readily be seen. Let us first suppose the n-equation to be a trinomial of one span. Let a root or pair of roots change continuously so that the span becomes less and less convergent, and finally not convergent but divergent: a new dominant has arisen, dividing the roots into two classes, k and $n - k$ in number respectively. Let now the latter span be again subdivided without destroying the dominant in question: the k roots of the other series remain together. As another supposition, let us for the sake of argument, imagine one and the same root to be in two adjacent spans, on both sides of a given dominant. Let all the other middle dominants gradually subside by changes of other roots until we have a new trinomial with the given root still on both sides of the middle dominant: the situation is absurd, and it is equally absurd to suppose any root to be either annihilated or created by gradual continuous change of other roots.

It is to be observed that coefficients are not to be counted as "dominant," in the sense here employed, unless they are relatively of larger size. It is possible that of three coefficients, a, b, c, we may find b relatively small, and that we may also find a convergent series for the span from b to c: if so, the order of magnitude will be disarranged. Cases of that sort are seldom likely to be observed, however, unless specially invented. Lagrange laid it down that the single root of the equation $x = \omega + \phi x$ developed by his method of 1768 (a method equivalent to that special case of the present general method in which $n = 1$) is always the smallest root; but his statement is not always correct unless the coefficient of x is a dominant, in the sense here used.*

* That Lagrange's proof concerning the smallest root is incomplete was shown by Chio, in a memoir presented to the Institute of France in 1846. Chio had however apparently no idea of "dominants." That the smallest root can easily be found when the coefficient of x is relatively very large is of course familiar.

Each of the paragraphs which follow, having attached to them numbers, from 1 to 11, relates to the example having the same number in the work on equations already mentioned.

[1]. $x^4 + 3x^3 + 7x^2 + 10x + 1 = 0$. The dominants here are uncertain. It cannot be claimed, as obvious, either that they are 1, 10, 1; 1, 7, 10, 1; or 1, 3, 10, 1. In these cases respectively the spans would be either a like cubic and a like linear; a like quadratic and two like linears; or a linear, a quadratic, and a linear, all with like signs. But any one of these combinations would indicate two real negative roots and two imaginary, one of the real roots being the smallest of the four. When the dominants are at all doubtful the equation is unfit and must be transformed. Let $x = y - 1$; then $y^4 - y^3 + 4y^2 + y - 4 = 0$. Spans, two quadratics, respectively like and unlike; whence y has two (larger) imaginary values and two (smaller) real values of opposite signs. From this equation in y we find approximately that the four values of x are, $-.4 \pm 2.2\sqrt{-1}$ and $-1 \pm .9$, showing that the largest values are imaginary, which rules out 1, 3, 10, 1 as the dominants, and that the real parts are negative, which rules out 1, 10, 1; leaving 1, 7, 10, 1 as correct. For the details, beginning with the first y-span, let $y = 2u$; then $u^2 = -1 + \frac{1}{2}u - \frac{1}{8}u^{-1} + \frac{1}{4}u^{-2} = -1 + 2\phi u$, where $\omega^2 = -1$, $\phi u = (4u - u^{-1} + 2u^{-2})/16$, and by (14)

$$
\begin{aligned}
u &= \omega + \omega^{-1}\phi\omega + \dots = \omega + \omega^{-1}(4\omega - \omega^{-1} + 2\omega^{-2})/16 \\
&= 5/16 + 18\omega/16 .
\end{aligned}
$$

Hence, for a rough approximation, $u = .3 \pm 1.1\sqrt{-1}$, so that $y = .6 \pm 2.2\sqrt{-1}$ or thereabouts, and $x = -.4 \pm 2.2\sqrt{-1}$. To approximate to the real values, by operating upon the second span, we have $y^2 = 1 + \frac{1}{4}(y^3 - y - y^4) = 1 + 2\phi y$, where $\phi y = \frac{1}{8}(y^3 - y - y^4)$. Hence $y = \omega + \omega^{-1}\phi\omega + \dots$, where $\omega = \pm 1$, or $y = \omega + \frac{1}{8}(\omega^2 - 1 - \omega^3) + \dots$, whence $y = \pm .9$, and $x = \pm .9 - 1$. A better transformation, if close results were desired, could be got by putting $y^{-1} = v + 1/16$.

[2]. $x^4 - 4x^3 - 3x + 23 = 0$. Dominants, 1, -4, 23; indicating a linear span, unlike, and a cubic, also unlike; largest root real positive, another real positive, two imaginary. For a fitter equation, take $x = 2(1 + u)/(1 - u)$, so that $77u^4 + 24u^3 + 234u^2 - 80u + 1 = 0$. Here the spans are, quadratic with like signs, and two linear, each unlike; hence two larger roots imaginary, two smaller real and positive. If $u = (234/77)^{\frac{1}{2}} y = 1.74y$, we shall have

$y^2 + .18y + 1 - .20y^{-1} = 0$, neglecting the term in y^{-2}, whence $y^2 = -1 + 2(-.09y + .10y^{-1})$. Here $\omega = \pm\sqrt{-1}$, and $y = -.19 + \omega = -.19 \pm \sqrt{-1}$, so that $u = -.33 \pm 1.75\sqrt{-1}$ or thereabouts. For the larger real root, let $u = 40z/117$, whence $z = 1 - .04z^3 - .03z^2 - .04z^{-1}$. In the second member we may replace z by $\omega = 1$, whence $z \doteq .89$, $u = .31$. For the smallest root, let $u = v/80$, and we find similarly $v = 1.04$, $u = .013$.

[3]. $2x^4 - 13x^2 + 10x - 19 = 0$. Dominants, 2, -13, -19: two quadratic spans, unlike and like respectively; hence two larger roots real with opposite signs, two smaller imaginary. Take $x^{-1} = 1 - y$, and again $y^{-1} = .1v + .9$, whence $x = (v + 9)/(v - 1)$. Then $v^4 - v^2 - 402v - 598 = 0$, which is to be solved. Here the spans are an unlike cubic and a like linear. The smallest root is real negative, the others are one real positive, two imaginary. For the cubic span, let $v = (402)^{\frac{1}{3}}z = 7.38z$, so that $z^4 - .018z^2 - z - .198 = 0$, and $z^3 = 1 + 3(.006z + .066z^{-1})$. The values of ω are 1 and $-\frac{1}{2} \pm .866\sqrt{-1}$, and the corresponding values of ω^2 are 1 and $-\frac{1}{2} \mp .866\sqrt{-1}$. We have here $z = \omega + \omega^{-2}(.006\omega + .066\omega^{-1}) = .066 + \omega + .006\omega^2$, and the values of z are 1.07 and $-.44 \pm .86\sqrt{-1}$, whence those of v are 7.9 and $-3.2 \pm 6.3\sqrt{-1}$. For the smallest root, let $v = 299u/201$, whence by the usual linear process $u = -1 - 3/640$ nearly, $v = -1.5$.

[4]. $x^5 + 2x^4 + x^3 - 4x^2 - 3x - 5 = 0$. A quintic equation with a full quintic span, two terminal dominants only, with unlike signs: one real positive root, four imaginary. To secure a fitter equation for solution, let us try $x = y - 1$, and again $y^{-1} = u + 1/6$, giving $3888u^5 + 2160u^3 - 684u^2 + 1521u - 365 = 0$. Here the dominants are 3888, 1521, -365, so that the spans are a like quartic and an unlike linear; whence the smallest root is real positive, and the other four imaginary. For the imaginary roots let $u = (1521/3888)^{\frac{1}{4}}z = .79z$, giving $z^5 + .89z^3 - .36z^2 + z - .30 = 0$. Then $z^4 = -1 + 4(-.22z^2 + .09z + .08z^{-1})$, and $z = \omega + \omega^{-3}(-.22\omega^2 + .09\omega + .08\omega^{-1}) = -.08 + \omega - .09\omega^2 + .22\omega^3$. Here either $\omega = .707(1 \pm \sqrt{-1})$, $\omega^2 = \pm\sqrt{-1}$, $\omega^3 = .707(-1 \pm \sqrt{-1})$; or else $\omega = .707(-1 \pm \sqrt{-1})$, $\omega^2 = \mp\sqrt{-1}$, $\omega^3 = .707(1 \pm \sqrt{-1})$. For the first pair, $z = .47 \pm .8\sqrt{-1}$, and for the second, $z = -.63 \pm .9\sqrt{-1}$. The corresponding values of u are $.37 \pm .6\sqrt{-1}$ and $-.49 \pm .7\sqrt{-1}$. For the smallest root, let $u = 365v/1521 = .24v$, whence $.01v^5 + .08v^3 - .11v^2 + v - 1 = 0$, and $v = 1.02$, $u = .24$.

[5]. $x^4 - 2x^3 - 7x^2 + 10x + 10 = 0$. Dominants, 1, -7, final 10: two quadratics, each with unlike signs, hence two pairs of real roots, larger and smaller, the roots of each pair having opposite signs. If we put $x = y + \frac{1}{2}$, so that $y^4 - 17y^2/2 + 2y + 209/16 = 0$, the same characteristics appear. Even this will not give close approximations without going further in the series than we have hitherto done. We have in fact not yet had occasion to take more than the first term beyond ω, a term which I shall hereafter, overlooking the ω, speak of as the "first term," so as to speak of the term containing $(\phi E_0)^m$ as the m^{th} term. If for another trial, we take $x = s + 1$, and then $s^{-1} = t + 1/8$, clearing of fractions by $t = y/8$, so that $x = (y + 9)/(y + 1)$, we have $3y^4 - 130y^2 + 8y + 1159 = 0$. Let us take this as the equation to be solved, though it is really not much fitter than the one before. The characteristics are the same as for the original equation. For the larger pair, let $y = (130/3)^{\frac{1}{2}} u = 6.58\,u$, and let us carry the work to two places. We have $u^4 - u^2 + .0093\,u + .2057 = 0$, whence $u^2 = 1 + 2\phi u = 1 + 2\,(-.0047\,u^{-1} - .1029\,u^{-2})$. Then $(\phi u)^2 = .001\,u^{-3} + .011\,u^{-4}$, $(\phi u)^3 = -.001\,u^{-6}$. To the first term inclusive, the formula (14) is $\omega + \omega^{-1}\phi\omega = \omega + \omega^{-1}(-.005\,\omega^{-1} - .103\,\omega^{-2}) = -.005 + .897\omega$, where $\omega = \pm 1$, $\omega^2 = 1$. The second term is $\frac{1}{2}(\phi E_0)^2\omega^{0-3}(0-1) = \frac{1}{2}(.001\,E_0^{-3} + .011\,E_0^{-4})\,\omega^{0-3}(0-1) = .0005\,\omega^{-6}(-4) + .0055\,\omega^{-7}(-5) = -.002 - .028\,\omega$. The third term is $\frac{1}{6}(\phi E_0)^3\omega^{0-5}(0-1)(0-3) = \frac{1}{6}(-.001\,E_0^{-6})\,\omega^{0-5}(0-1)(0-3) = -.001/6\,\omega^{-11}(-7)(-9) = -.011\,\omega$. The second term gave us $.028\,\omega$, and the third now gives us $.011\omega$, both with the same negative sign, which must affect all further terms. As the convergency is not rapid, we may add $-.02\omega$ as probably covering the succeeding terms. Summing, $u = -.01 + .84\omega$, the two values of u being .83 and $-.85$, and from these the values of y are 5.5 and -5.6. For the remaining values of y, which must be smaller, let $y = (1159/130)^{\frac{1}{2}} v = 2.986\,v$, so that the equation becomes $.206\,v^4 - v^2 + .0206\,v + 1 = 0$, whence $v^2 = 1 + 2\,(.010\,v + .103\,v^4)$. Then $(\phi v)^2 = .002v^5 + .011\,v^6$, and $(\phi v)^3 = .001\,v^{10}$. The formula (14) gives $v = \omega + \omega^{-1}(.010\omega + .103\,\omega^4) + .004\omega^2 + .028\omega^3 + .011\omega^5 + \ldots$ Making as before a slight allowance for the terms neglected, again all of the same sign, say $+.02\omega$, we have $v = .014 + 1.16\omega = .014 \pm 1.16$, and $y = 3v = .05 \pm 3.5$, so that there is a positive root somewhat greater than 3.5 and a negative root somewhat less. This equation, $3y^4 - 130y^2 + 8y + 1159 = 0$, may be resolved into the factors $3y^2 + 6y - 61 = 0$ and $y^2 - 2y - 19 = 0$.

[6]. $x^5 + 3x^4 + 2x^3 - 3x^2 - 2x - 2 = 0$. Nothing here is obviously domi-

nant. Let $x = y - 1$, so that $y^5 - 2y^4 - y^2 + 3y - 3 = 0$. For a fitter equation, let us take $y^{-1} = v + 1/5$; then $375v^5 - 25v^3 + 15v^2 + 256v - 1862/25 = 0$. This shows a like quartic and an unlike linear: smallest root real positive, the rest imaginary. For the imaginary roots, take $v = \frac{4}{5}(\frac{5}{3})^{\frac{1}{4}}u = .91u$; then $u^4 - (\frac{5}{3})^{\frac{1}{2}}u^2/16 + 3(\frac{5}{3})^{\frac{1}{4}}u/64 + 1 - 931(\frac{5}{3})^{-\frac{1}{4}}u^{-1}/2560 = 0$, and $u^4 = -1 + 4(.020u^2 - .015u + .080u^{-1})$. Hence $u = \omega + \omega^{-3}(.020\omega^2 - .015\omega + .080\omega^{-1}) = \omega - .080 + .015\omega^2 - .020\omega^3$, where $\omega^4 = -1$, and the values of ω and its powers are the same as in the paragraph numbered 4. Hence the imaginary values of u are approximately $.64 \pm .71\sqrt{-1}$ and $-.80 \pm .68\sqrt{-1}$, and those of v are $.58 \pm .64\sqrt{-1}$ and $-.73 \pm .62\sqrt{-1}$. For the small real root, take $v = 931z/3200 = .291z$, so that $z = 1 - .017z^2 + .008z^3 - .010z^5$. Hence $z = .98$ and $v = .29$.

[7]. $x^3 + 11x^2 - 102x + 181 = 0$. Dominants, 1, -102, 181: spans quadratic and linear, both unlike, hence three real roots, one of them negative, the smallest being positive. This is merely a form of the more celebrated equation $x^3 - 7x + 7 = 0$, to which it is reduced by writing $(x+3)^{-1} + 3$ for x: The latter has the same characteristics, but may be made fitter by putting $x = 3y/2$, and $y^{-1} = 1 - \frac{1}{2}z^{-1}$, so that $x = 3z/(2z - 1)$ and $z^3 - 21z + 7 = 0$, again with the smallest root positive. To find the two larger roots, let $z = 21^{\frac{1}{2}}v = 4.583v$, whence $v^3 - v + .07274 = 0$, and $v^2 = 1 + 2(-.036)v^{-1}$. Taking up to the second term from the trinomial formula (2), we have $v = \omega - .036\omega^{-2} - .002\omega = -.036 + .998\omega$. As $\omega = \pm 1$, the values of v are -1.034 and $+.962$, whence the two larger values of z are -4.74 and $+4.41$. For the smallest root, let $z = \frac{1}{3}u$, whence $u = 1 + u^3/196$, and $u = 1.005$, $z = .335$.

[8]. $x^5 + x^4 + x^3 - 2x^2 + 2x - 1 = 0$. This may be a quintic with a quintic span, hence one real positive root, four imaginary. It is proper at this point to explain why there may be a dominant between the two at the ends. There are several considerations involved. As a rule, a variation of signs among the larger coefficients is favorable, because an accumulation of terms having the same sign is unfavorable to convergence. The most important point to examine, of course, is the absolute size of any doubtful coefficient. Other things equal, it is an unfavorable circumstance if any such coefficient is so large that it would be a dominant were the equation a trinomial; and we may therefore make a tentative use of the criterion of convergency for trinomials. To examine this case, let us suppose it reduced first to $x^5 - 2x^2 - 1 = 0$. Here $a = +\frac{2}{5}$, and the

case as a trinomial is convergent if, numerically, $a^n < k^{-k}(n-k)^{k-n}$, where $n=5$, $k=3$. We see that $2^5 5^{-5}$ is not less, but greater than $3^{-3}2^{-2}$. Next, suppose it reduced to $x^5+2x-1=0$. Here $a=-\frac{2}{5}$, $k=4$, and we see that $2^5 5^{-5} > 4^{-4}$. Both tests are contrary to convergence, and, notwithstanding the favorable unlikeness of the signs of -2 and 2, we may therefore suspect that the given equation contains two spans, a like quartic and an unlike linear. It may, however, be mentioned that a first approximation attempted from the quintic span gives $x=.68$, not far from exact. To obtain a fitter equation, take $x^{-1}=u+2/5$, so that $3125u^5+1250u^3+375u^2-3825u-4603=0$, this time undoubtedly a quintic span. Let $u=(4603/3125)^{1/5}v=1.08v$ nearly, so that $v^5=1+5(.18-.02v^2-.07v^3)$. Then $v=\omega+\omega^{-4}(.18-.02\omega^2-.07\omega^3)+\ldots\ldots$, where $\omega^5=1$. Employing the powers of ω listed at the end of this paper, we find, for the real root, $v=1.09$, whence $u=1.17$; for one pair of imaginary roots, $v=.36\pm1.2\sqrt{-1}$, and for the other pair, $v=-.91\pm.7\sqrt{-1}$, whence $u=.39\pm1.3\sqrt{-1}$, $u=-.98\pm.8\sqrt{-1}$.

[9]. $x^6-6x^5-30x^2+12x-9=0$. Dominants, $1, -6, -30, -9$; spans, an unlike linear, a like cubic, and a like quadratic. The largest root is real positive; the three next in size include one real negative and two imaginary having their real parts positive; the smallest pair is imaginary. For the large positive root, let $x=6u$, so that $u=1+5u^{-3}/216-u^{-4}/648+u^{-5}/5184$, whence without further approximation $u=1.02$, $x=6.12$. For the next three roots, let $x=5^{1/3}v=1.71v$, so that $v^3=-1+3(.10v^4+.08v^{-1}-.03v^{-2})$, whence $v=\omega+\omega^{-2}(.10\omega^4+.08\omega^{-1}-.03\omega^{-2})=-.08+\omega+.07\omega^2$. For $\omega=-1$, this gives $v=-1.01$ for the real root, and $x=-1.72$. For $\omega=\frac{1}{2}\pm.866\sqrt{-1}$, $\omega^2=-\frac{1}{2}\pm.866\sqrt{-1}$, it gives $v=.38\pm.92\sqrt{-1}$, $x=.6\pm1.6\sqrt{-1}$. For the two smaller imaginary roots, let $x=(3/10)^{1/3}z=.548z$, so that $z^2=-1+2(.365z+.016z^5-.001z^6)$ or say $z^2=-1+2(.4z)$, whence $z=\omega+\omega^{-1}(.4\omega)=.4+\omega=.4\pm\sqrt{-1}$, and $x=.2\pm.5\sqrt{-1}$.

[10]. $2x^6-18x^5+60x^4-120x^3-30x^2+18x-5=0$. Dominants, $2, -18, -120, -5$: spans, an unlike linear, a like quadratic, and a like cubic. The largest root is therefore real positive, the next two in size are imaginary, and of the remaining three smaller roots, one is real negative and the other two imaginary with real parts positive. For the large positive root, let $x=9y$, so that $y=1-.37y^{-1}$, the rest being unimportant, whence $y=.63$, $x=5.6+$. For the next pair, let $x=(20/3)^{1/2}u=2.58u$, so that, disregarding the two final

terms, we have $u^2 = -1 + 2(.1u^3 + .7u - .1u^{-1})$, $u = \omega + \omega^{-1}(.1\omega^3 + .7\omega - .1\omega^{-1}) = .7 + \omega = .7 \pm \sqrt{-1}$, giving $x = 1.8 \pm 3\sqrt{-1}$. For the three smaller roots, let $x = 24^{-1/3}v = .347v$, so that, neglecting the two first terms, $v^3 = -1 + 3(.4v - .2v^2 + .1v^4)$, $v = \omega + \omega^{-2}(.4\omega - .2\omega^2 + .1\omega^4) = -.2 + \omega - .3\omega^2$. For the real root, $\omega = -1$, $v = -1.5$, $x = -.5$. For the imaginary, $\omega = .5 \pm .866\sqrt{-1}$, $\omega^2 = -.5 \pm .866\sqrt{-1}$, and $v = .5 \pm .6\sqrt{-1}$, $x = .2 \pm .2\sqrt{-1}$.

[11]. $2x^3 + 15x^2 - 84x - 190 = 0$. Dominants, 2, -84, -190: a quadratic span, with unlike signs, and a linear, with like signs. There are therefore three real roots, two of them negative and one of these the smallest, the third positive. If $x = y - 5/2$, we have the equation heretofore fully discussed as (3).

Lest the reader may suppose that, in the examples above illustrated, unusual means have been employed to secure specially favorable transformations, it is proper to remark that, while such unusual means ought to be discovered for the most perfect working of the method, none worth mentioning has yet been devised. The transformations used have, with one exception, been of the simplest, as the following list will show. In this list the word "direct" means "direct suppression" of the second term, and "reverse" means "reverse suppression" of the term next to the last. Newton's equation, reverse; Murphy's, none; Jelinek's, none; equation (3), none; No. 1, $x = y - 1$; No. 2, $x = 2y$, $y = (1 + u)/(1 - u)$; No. 3, $x^{-1} = 1 - y$, and reverse; No. 4, $x = y - 1$, and reverse; No. 5, first transformation, direct; second, $x = s + 1$, and reverse; No. 6, $x = y - 1$, and reverse; No. 7, from the usual form, $x = \frac{3}{2}y$, and reverse; No. 8, reverse; No. 9, none; No. 10, none; No. 11, direct. I am able to make only one suggestion, illustrated by No. 2. If an equation $f(x) = 0$ can be brought to the form in which $f(1) = \pm 1$, the transformation $x = (1 + u)/(1 - u)$ may be valuable. No. 2 is $x^4 - 4x^3 - 3x + 23 = 0$, and it is put into the form in question by taking $x = 2y$, whence $16y^4 - 32y^3 - 6y + 23 = 0$.

The present method applies equally well to the computation of the roots of equations with imaginary coefficients, coefficients which are regarded as having respectively the size of their moduli. Such equations are of greater generality than the ordinary, in which the imaginary roots are restricted to going in pairs. For a simple example, take $x^3 - x\sqrt{-1} - 1 = 0$. Here the span is cubic, $\omega^n = 1$, $a = \frac{1}{3}\sqrt{-1}$, and the formula from (2) is $x = \omega + \omega^2 a - \frac{1}{3}\omega a^3 + \frac{1}{3}\omega^2 a^4 + \dots$. Here $a^3 = -a/9$, $a^4 = 1/81$, so that $x = \omega + \omega^2/243 + a(\omega^2 + \omega/27)$, disregarding the subsequent terms. As either $\omega = 1$, $\omega^2 = 1$, or else $\omega = -\frac{1}{2} \pm \sqrt{-3}$,

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$\omega^2 = -\frac{1}{2} \mp \sqrt{-3}$, we find directly the three values of x, namely, $1.004 + .346\sqrt{-1}$, $-.225 + .690\sqrt{-1}$, and $-.779 - 1.036\sqrt{-1}$, which vary but slightly, if at all, from the accurate values. When the independent term ω^n is imaginary, the values of ω may be found by the usual method for computing the n^{th} roots of an imaginary quantity by the aid of De Moivre's theorem.

That difficulties will arise when we attempt to apply the formula to cases in which there are no obvious dominants is certain. The case of equal roots has already been mentioned as of that nature. Equal roots may be regarded as on the line between the real and the imaginary, and so we may suppose that an equation containing them may, as to its characteristics, be exactly on the dividing line between one set of dominants and another. Even a probably convergent case may present difficulties when two roots are equal: for example, $x^5 + x^4 - x - 1 = 0$. Treating this as a quintic span, we must accept the sum of two of the four imaginary series produced (if they can be summed, as seems probable), as equal to zero. To illustrate another class of doubtful cases, in which one or more of the series is divergent, let us take $x^4 + x^3 - x^2 + x + 1 = 0$. If we treat this as a quartic span we shall derive for the imaginary roots $.65 \pm .76\sqrt{-1}$, but for the real roots we shall obtain, instead of $-1.15 \pm .57$, the unintelligible result $-1.15 - \infty\sqrt{-1}$. If the sign of x be changed and 1 substituted for x we have as the sum of the terms -1, and we may put $x = -(1+u)/(1-u) = (u+1)/(u-1)$, producing $3u^4 + 14u^2 - 1 = 0$, an equation eminently fit. In treating this equation as a quartic span we were in fact assuming it to have four imaginary roots. The result shows that we should look upon it as having spans either linear and cubic; linear, quadratic and linear, or cubic and linear.

Having thus a method for the easy determination of the values of the roots, real and imaginary, not to speak of the preliminary recognition of their nature by inspection, there seems to be no further need of those various methods for distinguishing the real roots and assigning limits to their values, of which Sturm's was the latest and best. Are we then to say that for computing more closely the value of a single real root we may also dispense with the orderly and satisfactory method which we owe to Horner? The answer may eventually be both yes and no. As the culmination of arithmetical processes, based on the same principles as other well-known and less general rules of procedure, such for instance as the extraction of the square root of a given number, it is not

supposable that Horner's method can ever be allowed to subside into obscurity. On the other hand, apart from the orderly and luminous character of the process itself, it has never been really proved to be intrinsically easier in actual work than that by series, due to Euler and Lagrange, a process which, as we have seen, is only a special case of the present general method, that case in which $n = 1$. It may be seriously questioned whether, for practical purposes, any one employing the present method for first acquaintance with the several roots will, unless for special reasons in special cases, abandon it for Horner's method when it comes to securing a closer approximation to the value of any one of them. To use the present method in such a case, having an approximate value a, it is only necessary to transform the equation by $x = y + a$, and then, having thus secured a linear span for the smallest root, make another application of the method by the use of formula (14). De Morgan once deprecated a suggested improvement of Horner's method, "considering that the process is one which no person will very often perform," since variations of known rules infrequently applied "afford greater assistance in forgetting the method than in abbreviating it." For a similar reason it may not unreasonably be assumed that the present method may probably, in practice, be followed in most cases to the end.*

Sometimes, indeed, the examples employed for illustrating Horner's method are such as the present method meets at once most satisfactorily, without requiring further treatment at all. For example, Todhunter's chief example, running over many pages, is $x^3 - 3x^2 - 2x + 5 = 0$, which, if $x = y + 1$ (direct suppression), is $y^3 - 5y + 1 = 0$, a quadratic and a linear span yielding for all the roots very convergent results by the use of (2). Again, the last example given by Burnside and Panton is $x^4 - 3x^2 + 75x - 10000 = 0$, as to which it is required to "find to three places of decimals the root situated between 9 and 10." The result is 9.886. If we put $x = 10y$, we have a quartic span, $y^4 = 1 + 4\phi y$, where $\phi y = .0075\,y^2 - .01875\,y$. Taking nothing beyond the second term of

*It will of course have been observed that, at least for determining the numerical value of ω when ω^n is not ± 1, or (what is the same) for transforming the equation to the form $\omega^n = \pm 1$, the present method is much facilitated by the use of logarithms. It is not inappropriate to quote from Burnside and Panton the quasi-prediction with which they close their book, at the end of a historical note on numerical equations: "Mathematicians may also invent, in process of time, some mode of calculation applicable to numerical equations analogous to the logarithmic calculation of simple roots. But at the present time the most perfect solution of Lagrange's problem is to be sought in a combination of the methods of Sturm and Horner."

the general series (14), we have readily, for the four values of y, .9886, -1.0261, and $.01875 \pm .9927\sqrt{-1}$.

For the elementary instruction of students it may be well to afford a first glimpse of the method of dominants by taking some simple case in which there are only two or three large coefficients, with the other coefficients relatively much smaller. For instance, the cubic $1000x^3 + x - 1000 = 0$ will be admitted by any one to have three roots not very far in value from the three cube roots of unity; or if it be disputed, the discrepancy can be increased till it be admitted. Then $x^3 + x^2 + 10000x - 1 = 0$ may be used in like manner. Any one can easily be made to see that this resembles, first, a quadratic, and secondly, a linear equation. For, if $x = 100y$, we have $y^2 + y/100 + 1 - y^{-1}/1000000 = 0$, in which the last term is plainly of little importance; and similarly, if $x = u/10000$, we have $u = 1 - u^2/100000000 - u^3/1000000000000$, which cannot differ much from $u = 1$.

A simple proof of the convergency-criterion stated for trinomials is had by regarding the quotient formed by dividing the general $(m+n)^{\text{th}}$ term by the m^{th}, and considering m increased indefinitely. This supplies the criterion for a series comprised of the terms numbered m, $m+n$, $m+2n$, and so on, and the general series comprises n series of that nature, all having the same criterion. We have seen, as regards trinomials in which the middle term is not a dominant, that all the roots can be found by a single convergent series; and when the middle term is a dominant, which means, when the middle term has any value other than those included in the first case, the n roots are broken up into two classes, k roots of the first class being found by one convergent series, and $n-k$ roots of the second class being found by a second convergent series. All this is proved by the aid of the criterion of convergency. That similar facts are true concerning equations having more than three terms is also reasonably clear, yet in the absence of a general convergency-criterion we have not the same sort of proof for such equations as we have for trinomials, that every equation must be composed of mutually exclusive spans for each of which the series is convergent.

That the nature of the roots, as to reality and relative size, is disclosed when fitness is established and the spans recognized, is sufficiently proved. Our inferences as to the signs of the roots depend on an assumption not yet proved, viz. that the sign of the sum of the series is the same as that of the ω with which the series begins. No such doubt exists concerning the distinction between real and

imaginary roots. The imaginary series always occur in pairs of equal value and of opposite signs. If, in any case wherein ω is imaginary, the sum of the imaginary terms is not zero, there are two imaginary roots, while if it is zero there are two equal roots. Notwithstanding the defect of proof as yet existing concerning the signs of the roots, the inferences derivable from recognized spans are most of them certain and all of them tentatively useful. When we subsequently ascertain by convergent series all the roots of the equation, the proof that the values so found are really the roots is absolute, and the correctness of the preliminary inferences made as to their nature is determined.

The following table shows the maximum value of a^n consistent with convergency for any trinomial coming within the limits of the table, and may be extended by reference to the criterion $a^n < k^{-k}(n-k)^{k-n}\omega^{nk}$. When ω^n has any other value than ± 1, the factor ω^{nk} must be supplied.

$n-k$.	$n=1$.	$n=2$.	$n=3$.	$n=4$.	$n=5$.	$n=6$.
8	$7^7.8^{-8}$	$6^6.8^{-8}$	$5^5.8^{-8}$	$4^4.8^{-8}$	$3^3.8^{-8}$	$2^2.8^{-8}$
7	$6^6.7^{-7}$	$5^5.7^{-7}$	$4^4.7^{-7}$	$3^3.7^{-7}$	$2^2.7^{-7}$	7^{-7}
6	$5^5.6^{-6}$	$4^4.6^{-6}$	$3^3.6^{-6}$	$2^2.6^{-6}$	6^{-6}	
5	$4^4.5^{-5}$	$3^3.5^{-5}$	$2^2.5^{-5}$	5^{-5}		5^{-5}
4	$3^3.4^{-4}$	$2^2.4^{-4}$	4^{-4}		4^{-4}	$4^{-4}.2^{-2}$
3	$2^2.3^{-3}$	3^{-3}		3^{-3}	$3^{-3}.2^{-2}$	$3^{-3}.3^{-3}$
2	2^{-2}		2^{-2}	$2^{-2}.2^{-2}$	$2^{-2}.3^{-3}$	$2^{-2}.4^{-4}$
1		1	2^{-2}	3^{-3}	4^{-4}	5^{-5}
0						
-1	2^{-2}	3^{-3}	4^{-4}	5^{-5}	6^{-6}	7^{-7}
-2	$2^2.3^{-3}$	$2^2.4^{-4}$	$2^2.5^{-5}$	$2^2.6^{-6}$	$2^2.7^{-7}$	$2^2.8^{-8}$
-3	$3^3.4^{-4}$	$3^3.5^{-5}$	$3^3.6^{-6}$	$3^3.7^{-7}$	$3^3.8^{-8}$	$3^3.9^{-9}$
-4	$4^4.5^{-5}$	$4^4.6^{-6}$	$4^4.7^{-7}$	$4^4.8^{-8}$	$4^4.9^{-9}$	$4^4.10^{-10}$
-5	$5^5.6^{-6}$	$5^5.7^{-7}$	$5^5.8^{-8}$	$5^5.9^{-9}$	$5^5.10^{-10}$	$5^5.11^{-11}$

The following list of certain roots of 1 and of -1, and of the powers of these roots, will be found useful for reference. If $\omega^2=1$, $\omega=\pm 1$. If $\omega^2=-1$, $\omega=\pm\sqrt{-1}$. If $\omega^3=1$, $\omega=1$, $\omega^2=1$, or else $\omega=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$, $\omega^2=-\frac{1}{2}\mp\frac{1}{2}\sqrt{-3}$. If $\omega^3=-1$, $\omega=-1$, $\omega^2=1$, or else $\omega=\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$, $\omega^2=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-3}$. If $\omega^4=1$, either $\omega=1$, $\omega^2=1$, $\omega^3=1$; or $\omega=-1$, $\omega^2=1$, $\omega^3=-1$; or $\omega=\pm\sqrt{-1}$, $\omega^2=-1$, $\omega^3=\mp\sqrt{-1}$. If $\omega^4=-1$, either $\omega=\sqrt{\frac{1}{2}}(1\pm\sqrt{-1})$, $\omega^2=\pm\sqrt{-1}$,

$\omega^3=\sqrt{\frac{1}{2}}(-1\pm\sqrt{-1})$; or $\omega=\sqrt{\frac{1}{2}}(-1\pm\sqrt{-1})$, $\omega^2=\mp\sqrt{-1}$, $\omega^3=\sqrt{\frac{1}{2}}(1\pm\sqrt{-1})$. If $\omega^5=1$, either $\omega=1$, $\omega^2=1$, $\omega^3=1$, $\omega^4=1$; or $\omega=.309\pm.951\sqrt{-1}$, $\omega^2=-.809\pm.587\sqrt{-1}$, $\omega^3=-.809\mp.587\sqrt{-1}$, $\omega^4=.309\mp.951\sqrt{-1}$; or $\omega=-.809\pm.587\sqrt{-1}$, $\omega^2=.309\mp.951\sqrt{-1}$, $\omega^3=.309\pm.951\sqrt{-1}$, $\omega^4=-.809\mp.587\sqrt{-1}$. (The fractions .309, .809, .587, .951, are given merely as approximations to the values of the exact expressions $\frac{1}{4}(\sqrt{5}-1)$, $\frac{1}{4}(\sqrt{5}+1)$, $\frac{1}{4}\sqrt{(10-2\sqrt{5})}$, and $\frac{1}{4}\sqrt{(10+2\sqrt{5})}$ respectively.) If $\omega^5=-1$, either $\omega=-1$, $\omega^2=1$, $\omega^3=-1$, $\omega^4=1$; or $\omega=-.309\pm.951\sqrt{-1}$, $\omega^2=-.809\mp.587\sqrt{-1}$, $\omega^3=.809\mp.587\sqrt{-1}$, $\omega^4=.309\pm.951\sqrt{-1}$; or $\omega=.809\pm.587\sqrt{-1}$, $\omega^2=.309\pm.951\sqrt{-1}$, $\omega^3=-.309\pm.951\sqrt{-1}$, $\omega^4=-.809\pm.587\sqrt{-1}$. If $\omega^6=-1$, see the example at the beginning. The list may readily be extended by the rules given in the text-books.